SOME COMMENTS ON THE USE AND ABUSE OF THE HODRICK-PRESCOTT FILTER

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1. Introduction

It is always good practice in scientific research to keep an eye on other, similar disciplines and try to improve one's own methods in relation to relevant advances in other areas. Many times, one discovers that isolation causes some very perverse effects on the development of science. As Keynes said in *The General Theory* (Keynes, 1936),

"*It is astonishing what foolish things one can temporarily believe if one thinks too long alone, particularly in economics (along with the other moral sciences), where it is often impossible to bring one's ideas to a conclusive test either formal or experimental*"

In this regard, there is no doubt that economists have made many important contributions to the methodology of time series analysis; contributions that are not only of significance to economics but also have relevance to other scientific disciplines where their influence has been considerable in the past. On some occasions, however, some economists either seem unaware of, or misinterpret, developments in other areas of mathematics and engineering. A case in point is part of the econometric literature in favour of, or in opposition to, the use of the Hodrick-Prescott filter (HP hereafter; Hodrick and Prescott, 1980; 1997), which seems to view the subject of signal filtering in a peculiarly myopic manner. Probably the best papers in favour of the HP filter are the HP reference itself and most of the Real Business Cycle literature (RBC; mainly all those deriving from the 'Minnesota School'). On the other hand, there have been quite a large number of detractors from such a point of view, one of the most representative criticisms being Cogley and Nason (1995; CN hereafter).

Within an economic context, the emergence of the HP filter and its attribution clearly derives from two major facts. First, HP used the filter for estimating (and so removing) long term trends from macro-economic time series. Second, HP specified a filter bandwidth (via a fixed value of the smoothing parameter or Lagrange multiplier λ)

in quarterly macro-economic time series. In a more general context, however, we are not sure why the HP filter is given its particular attribution, particularly since it was not the first use of such a filter and the HP paper took about 17 years to appear in the open literature.

Certainly, the basic idea behind this particular 'smoothing' filter was certainly not originated by HP but was derived from much earlier work by others. For example, an early proposal to solve the data smoothing problem in the same general manner as HP was by Whittaker (1923). Later, Stigler (1978) pointed out that a similar algorithm was used by actuarial scientists in the 1920's; and von Neuman apparently used it in the ballistics literature in the 1940's, as acknowledged by Prescott (see Kydland and Prescott, 1990; Hodrick and Prescott, 1997). In addition, the whole approach can also be considered from the standpoint of spline fitting (e.g. Reinsch, 1967); while, more recently still, Schiller (1973) and Akaike (1980) have developed similar approaches within a Bayesian statistical setting.

In more general terms, it has been shown (Jakeman and Young, 1979, 1984; Young, 1991; Young and Pedregal, 1996) that *exactly* equivalent smoothing results to those obtained from the HP filter (and its relatives) can be obtained by posing the smoothing problem in stochastic State Space (SS) terms and using a recursive Fixed Interval Smoothing (FIS) solution to the problem. This approach was also proposed by other researchers in the early 1980's, such as Brotherton and Gersch (1981). And it underlies the later explosion of research on the use of SS methods for forecasting and signal extraction applied to Unobserved Component (UC) and 'Structural' models of nonstationary time series (e.g. Harvey, 1984,1989; Young *et al.*, 1989. Hodrick and Prescott 1997) acknowledge these similarities, but without any reference to the smoothing part of the algorithms.

The purpose of the present paper is not only to draw attention to what we believe are some misleading conclusions in part of the research publications that have favoured the HP filter, but also to point out other rather misleading conclusions found in some papers that have criticised it. Let there be no doubt that, like CN, we deplore the uncritical use of the HP filter, which is better viewed as a rather limited version of a more general class of smoothing filters (see below and Young and Pedregal, 1996). We agree entirely that it can yield misleading results if used uncritically. But, unlike CN, we are not prepared to reject such simple smoothing filters out-of-hand and so 'throw the baby away with the bath water'. If used with discretion, there is no doubt that they yield extremely fast results and can prove very useful for the initial, exploratory analysis of time series. Indeed, the FIS equivalent of the HP filter (but with the sensible addition of a user-specified, rather than fixed, bandwidth) has been available for more than twenty five years in our microCAPTAIN computer program[1]. Here, it is used very effectively for just this kind of exploratory analysis, with the user specified smoothing parameter carefully defined in relation to the spectral properties of the data (e.g. the periodogram or, preferably, the autoregressive (AR) spectrum).

2. General Comments

The HP filter can be interpreted in least squares optimisation terms, with the smoothness introduced by the addition of a Lagrange multiplier term that penalises the second difference of the estimated trend. But this approach, which is termed 'regularisation' in the numerical analysis literature, is not the only way of formulating this smoothing problem. For example, it has been pointed out (Jakeman and Young, 1979, 1984; Young, 1991; Young and Pedregal, 1996) that deterministic regularisation is the *exact* equivalent of recursive Kalman filter/FIS estimation based on a stochastic SS formulation of the

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[1] A new version of this MS-DOS program is the multi-platform, CAPTAIN Time Series Analysis and Forecasting Toolbox in Matlab. Information about this software is available in http://cres1.lancs.ac.uk/captain/ and a beta-test version is available from the second author.

problem. Moreover, the stochastically defined equivalent of the HP filter (which is termed IRWSMOOTH in microCAPTAIN) is obtained quite simply when the trend is modelled as an Integrated Random Walk (IRW) process[2]. Here, the 'Noise Variance Ratio' (NVR), defined as the ratio of the variances of the white noise input to the IRW process and the assumed white observation noise, serves *exactly* the same smoothing function as the *inverse* of the Lagrange multiplier λ in the regularising functional.

This complementary formulation of the smoothing problem is established most transparently by reference to traditional Wiener-Kolmogorov filter theory, which shows that the low frequency estimate of the trend $\hat{T}_t$ is related to the input time series y_t by the equation (see e.g. Young, 1994; Young and Pedregal, 1996),

$$\hat{T}_t = \frac{NVR}{NVR + (1-L)^2(1+L^{-1})^2} y_t \qquad (1)$$

where L is the lag (backward shift) operator. From this result, it is easy to demonstrate some extremely interesting, and perhaps rather surprising, properties of HP-IRWSMOOTH derived trends. For instance, Pedregal (1995) and Young and Pedregal (1996) show that the fourth difference of the estimated trend is *exactly* equal to the detrended series, lagged by two samples and re-scaled by a factor that is *exactly* equal to the *NVR* parameter (equivalently $1/\lambda$) used in the FIS estimation! And since FIS estimation is *exactly* equivalent to regularisation, Wiener-Kolmogorov smoothing and cubic smoothing spline estimation, this result also applies equally to these other methods.

The link between the non-recursive, deterministic regularisation and recursive, stochastic FIS estimation has most important connotations. In particular, we believe that SS

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[2] Harvey (1989) would refer to this as a Local Linear Trend model with the variance of the noise on the trend state constrained to zero. However, the IRW terminology has been in use for a

formulation of FIS estimation has many theoretical and practical advantages over the regularisation approach. Probably the main one is its flexibility, since the SS formulation facilitates the inclusion of more and diverse components into the model (such as seasonal components, influences of exogenous variables via regression variables or transfer functions, multivariable models, etc.). It is also computationally more efficient and allows inherently for aberrations in the data such as missing observations or gaps and abrupt changes in level or slope (see Young and Ng, 1989). Moreover, the recursive formulation is ideal for adaptive forecasting and backcasting outside the data, as well as on-line utilisation, in applications such as adaptive signal processing, forecasting and control (e.g. Young *et al*, 1999). And the FIS algorithms can also be extended easily to detect and allow for outliers, including automatic 'robustness' modifications, which are particularly simple in recursive processing. Such extensions are not so natural and, where feasible, they are much more complicated to implement in the regularisation approach (see e.g. Akaike, 1980).

Finally the stochastic formulation of the SS formulation allows for two additional advantages. First, it means that it is easy to optimise 'hyper-parameters', such as the smoothing parameter NVR (or $1/\lambda$), using maximum likelihood methods based on *prediction error decomposition* (Schweppe, 1965) or some equivalent method. Secondly, under the Gaussian assumptions inherent in the standard SS approach, the statistical properties of the extracted components are provided by the FIS algorithm, so that standard error bounds are a natural consequence of the analysis. These are not provided by the 'deterministic' HP regularisation algorithm.

3. Spurious Cycles and Artificial Co-Movements

The HP filter has been applied at least in two main ways. Firstly, it has been used as an alternative to simple differencing for inducing level-stationarity in economic time series.

Secondly, it has been used as a means to characterising the business cycle and to detect co-movements in time series. In the former case, it is well known that differencing, promoted by Box and Jenkins (1970) as a way of removing stochastic trends in the time series, results in an amplification of the high frequency components of the signal. On the other hand, the HP-IRWSMOOTH filter extracts the trend with specified attenuation (depending on the λ or NVR parameter) of the remaining spectral information. These two ways of removing the secular trend in time series have encouraged the development of a theoretical distinction between Trend Stationary (TS) and Difference Stationary (DS) series.

There are a number of specific criticisms made by most researchers who do not favour the use of the HP-IRWSMOOTH filter. Perhaps the most controversial point, stressed for example by CN, is the warning that possible artificial 'co-movements' may be detected in USA macroeconomic time series due to spurious cycles created by the filter. The main line of this argument is that the spectrum of the remaining 'cyclical' component $y_t - \hat{T}_t$, as obtained after detrending the data by the HP filter, includes an 'artificial' spectral peak that is related more to the properties of the filter than those of the series itself[3]. They then point out that the cyclical-like behaviour arising in the de-trended series from the presence of this spectral peak can lead to the detection of artificial co-movements in similarly analysed time series.

CN seek to demonstrate the possibility of such spurious co-movements among variables by examining the Cross Correlation Functions (CCF's) between filtered variables in a number of real and simulated data examples. It is certainly true that any signal processing filter, by definition, attenuates the power in certain parts of the frequency

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[3] Note that the use of the difference operator as a device to remove trends has a similar problem, i.e. it removes the trend but induces amplification of high frequency cycles that may not be of importance to the primary purposes of the analysis.

spectrum and so effectively enhances the power at other parts of the spectrum. But it is also equally true that the uncritical use of CCF's to infer relationships between the filtered variables obtained in this manner can itself lead to misleading conclusions, as we demonstrate later.

Of course, it is entirely correct for CN to conclude that cycles formed by the smoothing of Random Walk (RW) simulated data are likely to be spurious. But *real* data rarely constitute an RW process and, if they do not, then any extracted cycles may well be explained in physically meaningful terms. For example, if the spectral properties of the 'actual' cycle are known, it is easy to define a value of λ that corresponds to the relevant band-pass for extracting this 'actual' cycle. Of course, it is unlikely that this value will be the standard λ value (1600) proposed by HP, except in the case of a business cycle in quarterly data. And it is clearly unlikely that, in these circumstances, a cycle so extracted will be 'spurious'. In any case, time series analysts are not likely to depend for their inferences on one specific method of signal processing alone. Surely most analysts will only accept an extracted cycle as being meaningful provided its presence is supported by other evidence and a properly conducted statistical analysis of the filtered data (see section 4 below) does not question the inference.

Another problem with the analysis of CN and other detractors of filtering approaches to signal extraction lies in their use (following HP) of a constant value for the smoothing parameter λ (1600) and, thereby, a fixed specification of the filter bandwidth. This fixed value may be appropriate for American macroeconomic quarterly series (as HP suggest) but it is clearly not appropriate in other applications not involving quarterly series. Indeed, if it is used with other data (e.g. monthly, annual etc.) then this fixed parameter filter will almost certainly yield spurious extracted cycles. But what time series analyst worth his salt would arbitrarily use an arbitrarily fixed bandwidth, low-pass filter in these circumstances?

4. Have Previous Critiques of Filtering Methods Been Fair and Valid?

Given the above comments, it is clearly important to consider further the arguments of those who stress the problem of spurious cycles and meaningless co-movements in filtered time series. In order to illustrate their argument about spurious co-movements, CN, for example, utilise both simulation and real examples. Relying on the spectra and CCF of detrended simulated series, they conclude that the extracted signals show co-movements and cross correlation at different lags from zero, even when such relationships do not exist. From these results, they conclude that the HP filter creates spurious co-movements that are more 'artifacts' of the analysis than 'stylised facts' about the series.

In order to illustrate our counter-arguments, we will utilise similar simulation results. In particular, we consider simulated data generated by the following model (as used by CN):

$$x_t = \frac{e_t}{\nabla} \qquad \begin{array}{l} y_{1t} = \dfrac{\xi_t}{\nabla} \\ y_{2t} = \dfrac{\eta_t}{\nabla} \quad \text{and} \quad \eta_t = e_t + \nu_t \end{array} \qquad (2)$$

where e_t , ξ_t , and ν_t are independent and identically distributed, $N(0,1)$, white noise processes and $\nabla = 1 - L$. In other words, x_t and y_{1t} are a pair of RW models whose associated white noise inputs are independent of each other; while the white noise inputs associated with y_{2t} and x_t are correlated at zero lag (the variance ratio of ν_t to e_t is 0.8).

Figs 1 and 2 present some interesting results obtained when the HP-IRWSMOOTH filter is applied to the series x_t ,y_{1t} and y_{2t} obtained from the simulation of (2) using the HP-favoured smoothing parameter λ=1600 (NVR=1/1600=0.000625). Fig.1 compares the three detrended series via time series and scatter plots; while Fig.2 shows the removes some common frequencies in both series, this does not imply that all the series show co-movements that relate to each other. In the time series and scatter plots of Fig.1, for instance, simple visual appraisal shows that the first pair of RW's are probably not correlated; whilst the second pair are obviously correlated at lag zero.

On the other hand, it is clear that the periodogram and CCF plots in Fig.2 can be used incorrectly, in the manner of CN, to suggest that there is a relationship between the detrended variables. In fact, such plots *only* show that the series have similar spectral properties; they do *not* show that the temporal properties and, therefore, the co-movements, are similar. The reason why the CCF plots can be highly misleading in this regard is that they are being used and interpreted incorrectly in statistical terms. Is it not well known that, because of the autocorrelation in the detrended, cyclical component introduced by the smoothing operation, the CCF must be considered with great care? In particular, the need to 'pre-whiten' autocorrelated time series prior to CCF analysis in time series modelling has been acknowledged for a long time and is an important aspect of the initial identification analysis suggested by Box and Jenkins (1970) for transfer function modelling. Only when the pre-whitening operation has been carried out does the CCF relate to the 'correct' relation between variables.

(INSERT FIGURES 1 AND 2)

In the present example, this is demonstrated well by Fig.3, which shows the CCF's subsequent to the pre-whitening of the detrended series[4]. Now, the truth is exposed: we see that, unlike the equivalent CCF plots in Fig.2, the detrended y_{2t} and x_t series are very significantly correlated at lag zero; and, as would be expected, the CCF sample estimate is close to the theoretical, simulated correlation of 0.8. On the other hand, again as expected, there is no statistically significant correlation indicated at all between the

[4] The pre-whitening filters where identified using standard tools, that suggested AR(2) models.

detrended y_{1t} and x_t, series. It is clear from these results that pre-whitening is *essential* in these circumstances. The spuriousness found by CN is, in fact, an artifact induced by the incorrectly applied CCF analysis. It is not, as they imply, a property of the detrended, cyclical components that could be considered, in any way, meaningful in real terms.

(INSERT FIGURE 3)

Similar results apply to real data. For example, figure 4 shows the detrended unemployment rate and GNP for the USA between the second quarters of 1948 and 1998 using again the HP-favoured smoothing parameter λ=1/1600 (NVR=0.000625), together with the sample CCF between the raw series without and with pre-whitening. A simple inspection of the figure shows that there is a clear negative relation between both series. From the CCF without pre-whitening, a simultaneous dynamic relation would be inferred; but a refined analysis, after pre-whitening, shows that, apart from the contemporaneous correlation, GNP *leads* unemployment.

Indeed, the CCF in figure 4 can be utilised as a precursor to dynamic modelling of the relationship between GNP and unemployment, since it is directly proportional to the impulse response of such a model (see e.g. Box and Jenkins, 1970, p. 379 *et seq*). Indeed, the efficacy of such CCF impulse response identification analysis in this example can be verified by direct discrete-time Transfer Function (TF) identification and estimation, based on the detrended data. This yields the following TF model:

$$y_t = \frac{-0.01}{1-0.467L}u_t + \frac{1-0.90L-0.08L^2}{1-1.60L+0.71L^2}e_t \qquad (3)$$

where y_t is the detrended unemployment series; u_t is the detrended GNP series; and e_t is a residual $N(0;0,08)$ white noise process. The impulse response of the deterministic part of this model (the first term on the right hand side of (3)) is plotted in figure 5 and, as expected, it is proportional (approximately) to the CCF in figure 4. The simulated TF model output, based only on the GNP input and the deterministic part of the TF model, explains 74% of the variance of the unemployment series; while the usual coefficient of determination based on the one-step ahead forecasting errors, using the complete stochastic model (3), is R^2=0.97.

(INSERT FIGURES 4 AND 5)

5. A Better Approach to Trend Estimation and Cycle Extraction

In the view of the present authors, it is possible to completely remove even the possibility of encountering spurious cycle/co-movement problems by modelling the trend and cyclical components (and any other components that may be present) simultaneously using an Unobserved Component (UC) model. Powerful UC modelling methodologies are now available and the number of references is immense, see e.g. Harvey (1989); Young *et al*. (1999) and references therein. These methods provide tools for the identification of economic cycles in time series *simultaneously* with any other components. In this way, the arbitrary definition of the cycle implicit in the HP filter may be tested against the data, and the problem of 'artificial peaks' induced by the filtering operation appearing in the spectra of the cyclical component is removed completely. Moreover, co-movements among the rest of components, such as the trends and seasonal components, may be tested as well. There are numerous examples of such analysis. For instance, Harvey and Koopman (1997) and Koopman *et al.* (1995), present examples of UC models used to analyse USA macroeconomic and other data; while Young *et* al. (1999) provide sophisticated tools for the identification and estimation of business cycles within a UC setting. An alternative and much more comprehensive treatment of USA macroeconomic data can be seen in Young and Pedregal (1997, 1999), where transfer functions models are introduced to show the relationships between medium term economic cycles in quarterly private capital investment, government spending and unemployment series.

6. Summary

In conclusion, the purpose of this short paper is to draw attention to certain rather misleading aspects of the literature that has appeared both in favour of and in opposition to the use of the Hodrick-Prescott filter. Like the authors who criticise the application of this filter, we would warn against its *uncritical* usage, as well as any related, overly-simple smoothing algorithms. But, unlike such authors, we feel that such data-filters (more correctly smoothing filters) can prove very useful if they are applied carefully and with adequate understanding of their inherent limitations, particularly in the early, exploratory stages of time series analysis. At the subsequent identification and estimation stages in the analysis, however, it is important to consider the low frequency trend as just one component in a complete UC model of the time series. As such, it should be estimated concurrently with any other identified components (cyclical, seasonal, etc.) using the more sophisticated fixed interval smoothing relatives of the simple smoothing filters that constitute an important element in the state space estimation of UC models. We believe that this is the currently most powerful approach to the extraction of trends and cycles from time series and will effectively remove the possibility of extracting ‘spurious’ cycles and detecting unreal co-movements among time series.

7. References

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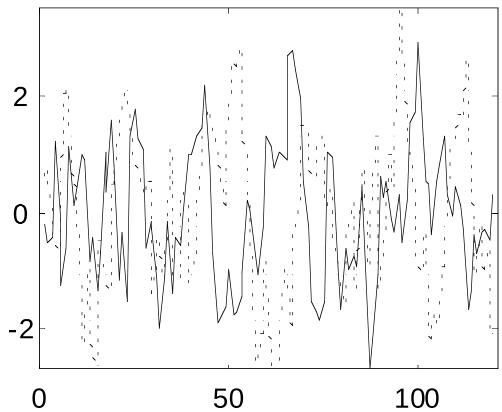
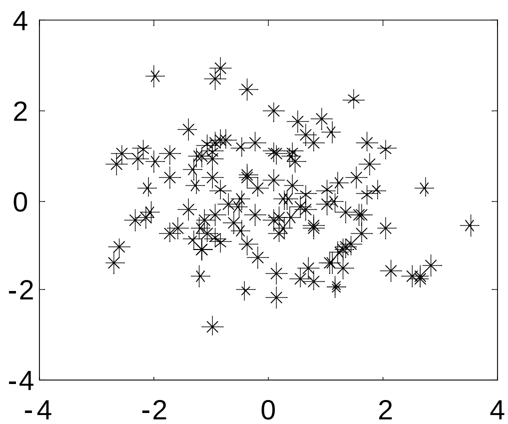
FIGURES

Detrended y_{1t} (dotted) and x_t (solid)



Detrended y_{2t} (dotted) and x_t (solid)

Scatter plot det. y_{1t} Vs det. x_t



Scatter plot det. y_{2t} Vs det. x_t

Figure 1: Simulation results: the detrended series and the associated scatter plots.

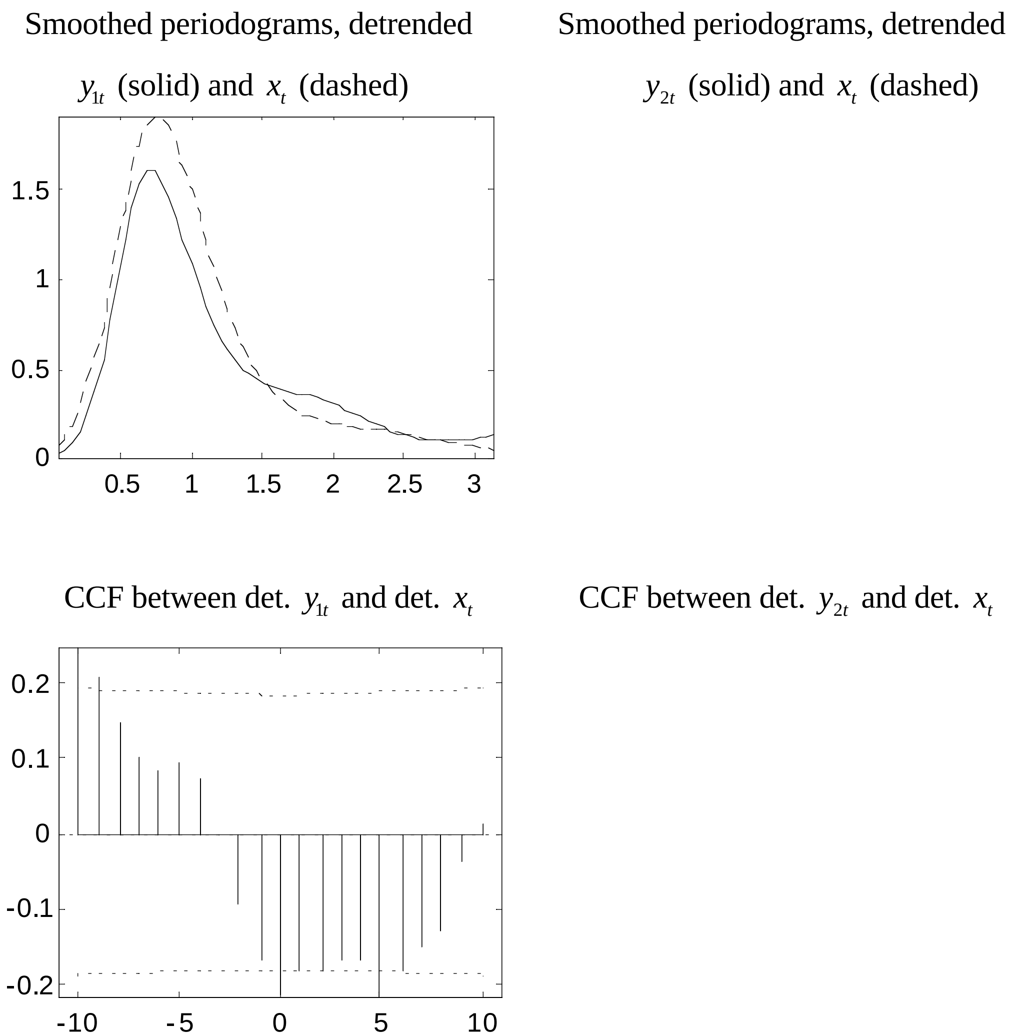


Figure 2: Simulation results: the periodogram and CCF plots for the detrended series.

CCF between det. y_{1t} and det. x_t CCF between det. y_{2t} and det. x_t

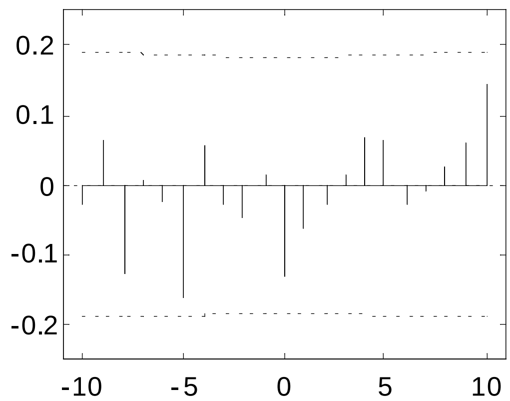


Figure 3: Simulation results: the CCF plots following pre-whitening.

Standardised detrended USA Unemployment rate and GNP. 1948(2)-1998(2)

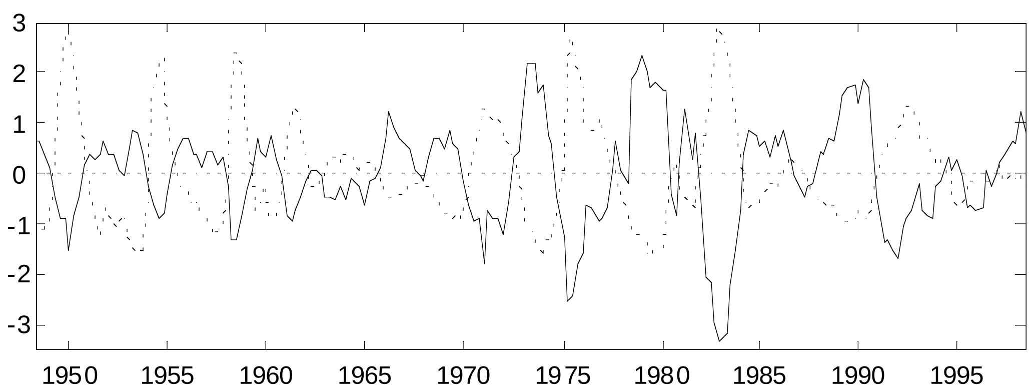


Figure 4: Detrended USA unemployment and GNP in a standardised scale between the second quarter of 1948 and 1998 (top plot) and the CCF between both series without pre-whitening (bottom left plot) and with pre-whitening (bottom right).

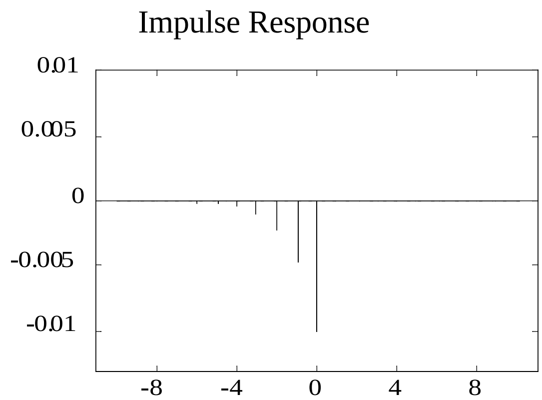


Figure 5: Impulse Response of deterministic part of model (3)